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A COMPARISON OF MAXIMUM LIKELIHOOD, EXPONENTIAL SMOOTHING AND BAYES FORECASTING PROCEDURES IN INVENTORY MODELING

by

Donald Gross
Robert Jay Craig

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This paper compares four major schemes used for forecasting mean demand to be used as input into an inventory model so that "optimum" stockage levels can be obtained. The inventory model is the classical order up to S, infinite horizon model with carry-over from period to period and complete backordering. Maximum likelihood, exponential smoothing, standard Bayes and adaptive Bayes schemes are used and results, via Monte Carlo simulation, are obtained for the total sum of discounted costs for (1) stationary demand, (2) long term trend and (3) shock changes in mean demand.

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I. INTRODUCTION

This paper compares four major schemes used for forecasting demand to be used as input to an inventory model so that "optimum" stockage levels can be obtained. Although the inventory model used (the classical periodic review, order up to S, infinite horizon model with complete backordering) assumes a known stationary demand distribution, it is seldom true that demand is in reality stationary. This study assumes demand is known to be Poisson, but the mean demand in each period is unknown. Maximum likelihood (sample mean), exponential smoothing, standard Bayes and adaptive Bayes forecasting schemes are compared, via Monte Carlo simulation in situations when the mean (1) is actually stationary, (2) has a long term trend and (3) incurs "shock" changes at random times. The criterion for comparison is the sum of the total discounted costs.

II. INVENTORY MODEL

Denoting the holding penalty by h (\$ per unit on shelf at the end of a period), the backorder penalty by π (\$ per unit in backorder at the end of a period, the unit purchase cost by c

($ per unit), the cumulative distribution function of the demand per period by $F(\cdot)$ and assuming no fixed cost of ordering and zero leadtime, the policy which minimizes the total sum of discounted costs over an infinite horizon with discounting factor α is given by (see, for example, [5], p. 386 or [6]),

"At the beginning of each period order

$$\max [S^* - z, 0] ,$$

where

z = on hand inventory before ordering

and

$$F(S^*) = [\pi - c(1-\alpha)]/(h+\pi) \text{ ." } \tag{1}$$

If $F(\cdot)$ is discrete, the equality in (1) becomes a "just greater or equal to" inequality, that is, S^* is the smallest integer value for which $F(S^*)$ is just greater or equal to the right hand side.

Further, it can also be shown that for a finite horizon of n periods, the optimal policy is to order up to a single critical value, S_n , where

$$S_1 \leq S_2 \leq \cdots \leq S_{n-1} \leq S_n < S^*$$

and S^* satisfies Equation (1). Practice has shown that convergence to S^* is quite rapid, especially when the cost of a unit, c , is relatively small.

Equation (1) is the expression used in the simulator to determine the best inventory level S^* for each period.

III. FORECASTING SCHEMES TESTED

It is assumed that the demand per period is Poisson, with mean λ , but that λ is unknown, and must be estimated from past data. Four major procedures (actually five procedures in all) are considered in this study: maximum likelihood, exponential smoothing (and exponential smoothing with trend), standard Bayes and adaptive Bayes. These are discussed in more detail below.

Maximum Likelihood (ML)

It is easy to show that the maximum likelihood estimator for the mean of a Poisson random variable is the arithmetic mean of all past observations (see, for example, [2], pp. 12-13). Thus, this scheme is extremely easy to use and furthermore would be the most familiar to those with only a passing acquaintance of statistics.

Exponential Smoothing (ES)

The simplest of the exponential smoothing routines is given as (see [2], p. 14, [1], p. 101 or [4], p. B-128)

$$\hat{\lambda}_n = ax_{n-1} + (1-a)\hat{\lambda}_{n-1} \tag{2}$$

where

$\hat{\lambda}_n$ = the estimated mean for period n

a = smoothing constant $(0 \le a \le 1)$[1]

x_n = the demand observed in period n .

In order to start the procedure, we use the initial condition

$$\hat{\lambda}_2 = (x_2+x_1)/2 \ ,$$

thus (2) is valid for $n \geq 3$.

Exponential Smoothing With Trend (EST)

If a long term trend is expected, better results are generally obtained by including a trend estimator explicitly in the forecasting scheme. The forecast, $\hat{\lambda}_n$, now becomes (see [2], pp. 14-15 or [4], p. B-128)

$$\hat{\lambda}_n = \tilde{S}_n + \tilde{T}_n \ , \tag{3}$$

---

[1]For all runs throughout this study, a value of 0.3 was used.

where

$$\tilde{S}_n = ax_n + (1-a)(\tilde{S}_{n-1}+\tilde{T}_{n-1})$$

$$\tilde{T}_n = b(\tilde{S}_n-\tilde{S}_{n-1}) + (1-b)\tilde{T}_{n-1}$$

b = trend smoothing constant $(0 \le b \le 1)$[2]

and the initial conditions used are

$$\tilde{T}_1 = 0$$

$$\tilde{S}_1 = x_1 ,$$

making (3) valid for $n \ge 2$.

Standard Bayes (SB)

For the Bayes schemes it is necessary to assume a prior distribution on λ , since under Bayesian analysis, λ is not assumed to be a fixed unknown constant but a random variable. The prior distribution assumed for λ is an exponential distribution with a known mean θ . The estimate $\hat{\lambda}_n$ is, for this standard Bayes analysis, the expected value of the posterior distribution for λ given the sum of the past $n - 1$ observations which we denote by t_{n-1} . The posterior distribution given t_{n-1} turns out to be gamma with parameters $n - 1 + 1/\theta$ and $t_{n-1} + 1$ (see [2], pp. 9-10) so that the expected value and hence $\hat{\lambda}_n$ is given by

(4) $$\hat{\lambda}_n = (t_{n-1}+1)/(n-1+1/\theta) .$$

Note that for $n = 1$ and $t_0 = 0$, $\hat{\lambda}_1$ is just the mean of the prior, namely θ , so that (4) holds for $n \ge 1$. Also, the limit of (4) as $n \to \infty$ is the arithmetic mean so that for large n , $\hat{\lambda}_n$ given by (4) is approximately equal to the maximum likelihood forecast.

---

[2]For all runs throughout this study, a value of 0.3 was used.

Adaptive Bayes (AB)

The adaptive Bayes procedure (see, for example, Zachs [7]) instead of using the expected value of the posterior distribution as the Poisson λ in the inventory equation (Equation (1)), actually utilizes the marginal distribution of demand for the nth period given the sum of the past $n - 1$ periods of demand. This distribution function is used for $F(\cdot)$ in (1) and S^* determined accordingly. Denoting this marginal distribution as $g_n(\cdot)$ and its cumulative distribution by $G_n(\cdot)$, then (1) becomes

$$(5) \qquad G_n(S^*) \geq [\pi - c(1-\alpha)]/(h+\pi) .$$

It turns out (see [2], pp. 10-12 or [7]) that $G_n(\cdot)$ is a negative binomial with parameters $t_{n-1}' + 1$ and $[1/(n+1/\theta)]$ so that

$$(6) \quad G_n(S^*) = \sum_{x=0}^{S^*} g_n(x|t_{n-1}) = \sum_{x=0}^{S^*} \binom{x+t_{n-1}}{x} \left(\frac{n-1+1/\theta}{n+1/\theta}\right)^{t_{n-1}+1} \left(\frac{1}{n+1/\theta}\right)^{x} .$$

IV. SIMULATOR

Poisson demand is generated via a Monte Carlo procedure with the mean λ being an input parameter. In addition, a value of θ is required as input in order to operate the Bayes schemes. The user has the option of requiring λ to remain constant (stationary case), having λ increase by any specified amount each period (long term trend case) or having λ "jump" by a specified amount at specified times (shock case). The trend and jump values are also input parameters.

The cost parameters (c, h, π) as well as the discount factor α and initial on hand inventory must also be inputted. The user also specifies the number of periods to be run and the number of times a given run is to be replicated (replications refer to keeping all input parameters fixed, but generating different sets of random deviates for the demand stream thus allowing for statistical analyses on the output). An initialization period must also be specified during which forecasts are made and stockage levels calculated but resulting costs are not included in the discounted cost streams, thereby removing start-up effects from the cost comparisons. The program also calls for the specifications of a pre-initialization period in which demand values are generated prior to calculating the first S^* so that every forecasting scheme will have the same number of observations prior to making its first forecast.

The user must specify which forecasting schemes are to be employed. All or any combinations of the five schemes discussed in Section III can be utilized. When more than one scheme is called for on a run set (a run set refers to the set of replications for a fixed set of input parameters) the same demand stream is used for each scheme. That is, demand is generated via Monte Carlo, the first scheme is used for forecasting and stockage calculations and costs are computed. Then, on the same set of demand, the next scheme is used, etc. Thus, one replication is generated for each scheme. After the last scheme is used, a new demand stream is generated and the process repeated to generate another replication for each scheme. The procedure continues until the desired number of replications are obtained. By using the same demand stream for each forecasting scheme on each replication, the cost differences are due solely to the effect of forecasting and hence the variation among the average discounted costs for the various forecasting schemes is kept as low as possible. Also, the cost results lend themselves nicely to statistical analysis by a paired-sample t test.

The simulator puts out various quantities of interest, the major being the average discounted expected costs per period, that is, the total sum of discounted expected costs for the number of periods run divided by the number of periods. Also, average on shelf inventory, average number in backorder and the demand stream generated as well as the resulting S^*'s calculated for each forecasting scheme are available. The simulator is written in FORTRAN IV and had been run on the IBM/360/50.

V. STATISTICAL TESTS

A chi-square goodness of fit test (see, for example, [3], pp. 285-6) was performed on the Poisson demand generator. A sample of 180 periods demand was generated with $\lambda = 5$. Nine class intervals were used and the calculated χ^2 test statistic was 8.6448. From χ^2 tables (see, for example, [3], p. 318), a 5% critical value for 8 degrees of freedom is 15.507 and a 10% critical value is 13.362. Thus, there is no evidence to reject the hypothesis that the generated data came from a Poisson distribution with $\lambda = 5$.

To determine if significant differences can be found in expected discounted costs when using different forecasting schemes, a paired-sample t test was employed (see, for example, [3], p. 269). This test utilizes the paired differences in the costs between any two schemes and tests if the average difference (over all replications of a given run set) is significantly different from zero.

VI. RESULTS

Prior to investigating the effect of forecasting schemes on inventory costs, two preliminary tests were made to determine (1) the sensitivity of the assumption on the prior mean (θ) in the Bayes schemes and (2) the effect of the discount factor (α). Next, three major cases were investigated to study the effect of forecasting schemes on inventory costs under (1) stationary demand, (2) long term trend and (3) shock changes in mean demand. Finally, an analysis of maximum likelihood versus the Bayes schemes for low mean demand was performed.

In performing the simulations, decisions must be made as to the set of costs used. These cost parameters are related to each other in the following way. The right hand side of Equation (1) (or (5)) is a particular fraction depending on the values of π, h and c , assuming α is fixed. Let this fraction be denoted by f , that is,

(7) $$f = [\pi - c(1-\alpha)]/(h+\pi) .$$

Equation (7) can be rewritten as

$$f = [\pi' - (1-\alpha)]/(h'+\pi') , \tag{8}$$

where

$$\pi' = \pi/c , \quad c > 0$$

$$h' = h/c , \quad c > 0 .$$

If $c = 0$, Equation (7) reduces to

$$f = \pi/(h+\pi) . \tag{9}$$

Since S^* depends solely on f, results are valid for all cost combinations yielding the same f. Thus, once it is decided on which values of f to consider for the simulation runs, the cost parameters can be determined.[3]

For the major runs, it was decided to investigate three values for f, namely, 0.8, 0.5 and 0.3. Table 1 shows the cost parameter settings used to yield these f values, for $\alpha = 1$ and 3/4 respectively.

TABLE 1

COST PARAMETERS FOR THREE COST RATIOS

	c = 1	
	$\alpha = 1$	
f	h	π
0.8	1	4
0.5	1	1
0.3	7	3
	$\alpha = 3/4$	
f	h	π
0.8	1	5.25
0.5	1	1.50
0.3	7	3.36

[3]The specific values used for the cost parameters do effect the total costs in a multiplicative way (assuming $c \neq 0$), but since we are interested in using these costs for comparative purposes only, we are free to choose any combination yielding the desired f.

Tables 2, 3 and 4 give the results of the test for sensitivity of the Bayes scheme to θ. Twenty replications of runs of 40 periods[4] each were performed for λ stationary at 5, $f = 0.8$ and $\alpha = 3/4$. Table 2 shows the mean and standard deviations of the average discounted costs. Table 3 gives the percent difference between $\theta = 5$ and 1 and $\theta = 5$ and 9 for each Bayes scheme, when the true mean is 5 and the demand process is stationary. The largest percent difference in Table 3 is less than 1.5%. Table 4 shows that when a paired t test is run on the differences, no statistical significance at either the 5% or 1% level is noted. Thus, for all other runs (except the final run sets) θ was always set equal to λ and it was felt that this would not provide a significant advantage to the Bayes schemes over the non-Bayes schemes.

Tables 5 and 6 give the results of run set 2 to determine the sensitivity to α assumptions. The cost ratio f was held at 0.8, while α and λ were set at 3/4 and 1, and 5 and 1 + 0.5 respectively (1 + 0.5 indicates λ starts at 1 and increases by 0.5 per period). Again, 20 replications of runs of 40 periods were performed. From Table 5 we see that the relative order of average cost per period is preserved across schemes for the different α's. Also, from Table 6, the percent differences are generally the same order of magnitude. It is noted, however, that when discounting takes place ($\alpha \neq 1$) a large reduction in the average cost per period results. This is due to the fact that the cost per period is essentially discounted to zero for the later periods. These later periods with zero or nearly zero period costs are averaged in with the earlier non-zero costs, resulting in a greatly reduced average period cost. Since this is effectively throwing out (or at least greatly diminishing the influence of) later period costs, it was concluded that a discount factor of one should be used which would allow all periods to be weighted equally. (By using an $\alpha = 1$, we are actually comparing the expected costs per period (cost

[4]These 40 periods are over and above 10 periods used for initialization plus 5 periods used for pre-initialization.

TABLE 2

RUN SET 1
SENSITIVITY OF BAYES FORECAST SCHEMES TO θ ASSUMPTIONS
$\lambda = 5$; $\alpha = 3/4$; $f = .8$
Average Discounted Cost Per Period

θ	SB		AB	
	MEAN	S.D.	MEAN	S.D.
1	0.845	0.031	0.844	0.031
5	0.837	0.030	0.834	0.026
9	0.833	0.028	0.834	0.030

TABLE 3

RUN SET 1
COMPARISONS OF PERCENT DIFFERENCES WITHIN A SCHEME
$\lambda = 5$; $\alpha = 3/4$; $f = .8$

θ PAIR \ SCHEME	SB	AB
1; 5	1.0	1.1
5; 9	-0.6	-0.1

TABLE 4

RUN SET 1
PAIRED t STATISTICS FOR DIFFERENCE IN θ
($t_{.05} = 2.093$, $t_{.01} = 2.861$)

θ PAIR \ SCHEME	SB	AB
1; 5	-0.796	-0.903
5; 9	-0.828	-0.115

TABLE 5

RUN SET 2
SENSITIVITY OF FORECAST SCHEMES TO α ASSUMPTIONS
$\lambda = 1,\ 1 + .5;\ \alpha = 1,\ 3/4;\ f = .8$
Average Discounted Cost Per Period

CASE \ SCHEME	ML		ES		EST		SB		AB	
	MEAN	S.D	MEAN	S.D.	MEAN	S.D.	MEAN	S.D.	MEAN	S.D.
$\lambda = 1;\ \alpha = 1$	2.612	0.065	2.734	0.082			2.615	0.064	2.615	0.064
$\alpha = 3/4$	0.272	0.024	0.275	0.022			0.272	0.024	0.272	0.024
$\lambda = 1+.5;\ \alpha = 1$	43.616	0.505			25.717	0.275	44.556	0.478	44.397	0.480
$\alpha = 3/4$	2.679	0.107			1.713	0.074	2.791	0.104	2.761	0.106

TABLE 6

RUN SET 2
COMPARISONS OF PERCENT DIFFERENCES BETWEEN SCHEME PAIRS

CASE \ SCHEME PAIR	AB;ML	AB;SB	AB;ES AB;EST	ML;SB	ML;ES ML;EST	SB;ES SB:EST
$\lambda = 1;\ \alpha = 1$	0.1	0	-4.5	-0.1	-4.6	-4.5
$\lambda = 1;\ \alpha = 3/4$	0.5	0	-0.9	-0.1	-1.1	-0.8
$\lambda = 1+.5;\ \alpha = 1$	1.8	-0.4	72.6	-2.2	69.6	73.2
$\lambda = 1+.5;\ \alpha = 3/4$	3.1	-1.1	61.2	-4.2	56.4	62.9

TABLE 7

RUN SET 3
STATIONARY DEMAND
$f = .8, .5, .3;\ \alpha = 1;\ \lambda = .5, 1., 5., 10., 15.$
Average Cost Per Period

CASE \ SCHEME	ML		ES		EST		SB		AB	
	MEAN	S.D.	MEAN	S.D.	MEAN	S.D.	MEAN	S.D.	MEAN	S.D.
$f = .8;\ \lambda = .5$	1.482	0.060	1.674	0.084	1.786	0.073	1.477	0.060	1.477	0.060
$\lambda = 1.$	2.484	0.071	2.654	0.078	2.730	0.080	2.480	0.067	2.476	0.068
$\lambda = 5.$	8.512	0.148	8.817	0.169	9.255	0.184	8.511	0.148	8.517	0.150
$\lambda = 10.$	14.882	0.229	15.131	0.250	15.561	0.256	14.882	0.229	14.890	0.229
$\lambda = 15.$	20.839	0.218	21.131	0.256	21.715	0.264	20.835	0.217	20.837	0.221
$f = .5;\ \lambda = .5$	0.935	0.049	1.071	0.048	1.075	0.045	0.936	0.049	0.937	0.049
$\lambda = 1.$	1.711	0.048	1.755	0.051	1.822	0.055	1.707	0.049	1.712	0.048
$\lambda = 5.$	6.891	0.104	7.022	0.117	7.222	0.126	6.892	0.104	6.892	0.104
$\lambda = 10.$	12.671	0.145	12.766	0.151	13.011	0.154	12.670	0.145	12.670	0.146
$\lambda = 15.$	18.072	0.165	18.162	0.177	18.467	0.196	18.075	0.165	18.079	0.165
$f = .3;\ \lambda = .5$	1.869	0.099	1.976	0.[illegible]0	2.285	0.149	1.869	0.099	1.869	0.099
$\lambda = 1.$	3.875	0.156	4.232	0.111	4.304	0.149	3.884	0.157	3.890	0.160
$\lambda = 5.$	12.361	0.237	12.829	0.269	13.577	0.305	12.361	0.237	12.346	0.235
$\lambda = 10.$	20.596	0.300	21.365	0.362	22.322	0.401	20.596	0.300	20.560	0.308
$\lambda = 15.$	28.117	0.340	28.715	0.414	29.962	0.514	28.117	0.340	28.065	0.339

TABLE 8

RUN SET 3

T STATISTICS ($t_{.05}=2.093$, $t_{.01}=2.861$)

CASE \ SCHEME PAIR	ML;ES	ML;EST	ML;SB	ML;AB	ES;EST	ES;SB	ES;AB	EST;SB	EST;AB	SB;AB
f = .8; λ = .5	-4.248**	-6.503**	1.0	1.0	-3.337**	4.504**	4.504**	6.759**	6.759**	0.0
λ = 1.	-4.363**	-5.726**	0.411	0.767	-2.463*	4.549**	4.559**	5.825**	5.759**	0.719
λ = 5.	-7.428**	-8.772**	0.568	-0.607	-6.997**	7.356**	7.457**	8.771**	8.764**	-0.737
λ = 10.	-2.924**	-6.107**	0.0	-0.670	-7.017**	2.924**	2.956**	6.107**	6.299**	-0.670
λ = 15.	-2.996**	-6.996**	1.371	0.089	-7.497**	3.010**	3.109**	6.986**	7.124**	-0.180
f = .5; λ = .5	-5.949**	-5.984**	-0.568	-1.000	-0.271	6.069**	6.077**	6.039**	6.022**	-1.000
λ = 1.	-2.928**	-5.231**	1.831	-1.0	-3.038**	3.442**	2.799**	5.553**	5.232**	-2.179*
λ = 5.	-3.694**	-6.779**	-1.0	-0.568	-7.574**	3.694**	3.645**	6.797**	6.732**	0.000
λ = 10.	-3.377**	-7.802**	0.568	0.370	-8.533**	3.409**	3.568**	7.791**	7.818**	0.0
λ = 15.	-1.964	-6.098**	-1.453	-1.750	-9.821**	1.886	1.823	6.018**	5.988**	-0.901
f = .3; λ = .5	-2.243*	-5.475**	0.0	0.0	-4.004**	2.243*	2.243*	5.475**	5.475**	0.0
λ = 1.	-2.763*	-2.784*	-1.677	-1.552	-0.961	2.710*	2.591*	2.734*	2.644*	-0.793
λ = 5.	-3.208**	-7.179**	0.0	0.649	-6.581**	3.208**	3.404**	7.179**	7.341**	0.649
λ = 10.	-5.308**	-8.255**	0.0	0.989	-8.127**	5.308**	4.873**	8.253**	7.806**	0.989
λ = 15.	-3.262**	-6.325**	0.0	1.789	-7.745**	3.262**	3.692**	6.325**	6.598**	1.789

* Significant at 5% level

** Significant at 1% level

TABLE 9

RUN SET 3
COMPARISON OF PERCENT DIFFERENCES IN AVERAGE PERIOD COSTS
FOR THOSE SCHEME PAIRS WITH SIGNIFICANT t STATISTICS

CASE \ SCHEME PAIR	ML;ES	ML;EST	ES;EST	ES;SB	ES;AB	EST;SB	EST;AB
$f = .8;\ \lambda = .5$	-12.9	-20.5	-6.7	13.3	13.3	20.9	20.9
$\lambda = 1.$	-6.8	-9.9	-2.8	7.0	7.2	10.1	10.3
$\lambda = 5.$	-3.6	-8.7	-5.0	3.6	3.5	8.7	8.7
$\lambda = 10.$	-1.7	-4.6	-2.8	1.7	1.6	4.6	4.5
$\lambda = 15.$	-1.4	-4.2	-2.8	1.4	1.4	4.2	4.2
$f = .5;\ \lambda = .5$	-14.6	-15.0	not sig.	14.4	14.3	14.6	14.7
$\lambda = 1.$	-2.6	-6.5	-3.8	2.8	2.5	6.7	6.4
$\lambda = 5.$	-1.9	-4.8	-2.8	1.9	1.9	4.8	4.8
$\lambda = 10.$	-0.8	-2.7	-1.9	0.8	0.8	2.7	2.7
$\lambda = 15.$	-0.5	-2.2	-1.7	not sig.	not sig.	2.2	2.1
$f = .3;\ \lambda = .5$	-5.8	-22.3	-15.6	5.8	5.8	22.3	22.3
$\lambda = 1.$	-9.2	-11.1	not sig.	9.0	8.8	10.8	10.6
$\lambda = 5.$	-3.8	-9.8	-5.8	3.8	3.9	9.8	10.0
$\lambda = 10.$	-3.7	-8.4	-4.5	3.7	3.9	8.4	8.6
$\lambda = 15.$	-2.1	-6.6	-4.3	1.4	2.3	6.6	6.8

TABLE 10

RUN SET 4
DEMAND WITH TREND
$f = .8, .5, .3;\ \alpha = 1;\ \lambda = 1 + .05,\ 1 + .1,\ 1 + .5$
Average Cost Per Period

SCHEME / CASE	ML		ES		EST		SB		AB	
	MEAN	S.D.	MEAN	S.D.	MEAN	S.D.	MEAN	S.D.	MEAN	S.D.
$f = .8;\ \lambda = 1+.05$	0.597	0.018	0.558	0.015	0.580	0.017	0.602	0.018	0.600	0.018
$\lambda = 1+.1$	0.958	0.017	0.802	0.014	0.826	0.013	0.971	0.019	0.967	0.019
$\lambda = 1+.5$	4.468	0.055	2.598	0.027	2.571	0.024	4.572	0.056	4.558	0.056
$f = .5;\ \lambda = 1+.05$	0.420	0.009	0.411	0.009	0.425	0.009	0.421	0.009	0.421	0.009
$\lambda = 1+.1$	0.678	0.010	0.628	0.009	0.642	0.010	0.682	0.010	0.683	0.010
$\lambda = 1+.5$	2.734	0.024	2.234	0.019	2.237	0.019	2.757	0.024	2.758	0.024
$f = .3;\ \lambda = 1+.05$	0.869	0.021	0.838	0.020	0.876	0.024	0.872	0.021	0.872	0.021
$\lambda = 1+.1$	1.345	0.022	1.183	0.023	1.227	0.027	1.353	0.022	1.353	0.022
$\lambda = 1+.5$	5.025	0.046	3.480	0.048	3.522	0.049	5.084	0.047	5.094	0.047

TABLE 11

RUN SET 4

T STATISTICS ($t_{.05}$=2.093, $t_{.01}$=2.861)

CASE \ SCHEME PAIR	ML;ES	ML;EST	ML;SB	ML:AB	ES;EST	ES;SB	ES;AB	EST;SB	EST;AB	SB;AB
f = .8; λ = 1+.05	5.833**	2.111*	-2.926**	-2.123*	-5.091**	6.211**	5.796**	-2.686*	-2.329*	1.761*
λ = 1+.1	10.787**	9.502**	-10.389**	-6.321**	-6.712**	11.705**	11.237**	-10.430**	-9.956**	3.332**
λ = 1+.5	46.454**	42.429**	-29.202**	-26.891**	2.994**	48.235**	47.208**	-44.209**	-43.370**	5.323**
f = .5; λ = 1+.05	2.901**	-1.498	-1.798	-2.483*	-5.910**	3.407**	3.635**	1.304	1.081	-2.042
λ = 1+.1	9.079**	5.785**	-5.252**	-5.467**	-10.105**	9.335**	9.595**	-6.143**	-6.348**	-1.831
λ = 1+.5	46.317**	37.729**	-31.064**	-29.437**	0.652	42.199**	42.089**	-39.522**	-39.475**	-3.327**
f = .3; λ = 1+.05	2.479*	-0.524	-2.189*	-2.470*	-4.196**	2.705*	2.775*	0.271	0.253	-0.224
λ = 1+.1	6.867**	4.559**	-4.780**	-4.746**	-3.957**	7.3[illegible]**	7.197**	-4.874**	-4.828**	0.084
λ = 1+.5	34.667**	29.056**	20.782**	-23.396**	-2.548**	35.[illegible]	35.729**	-29.601**	-30.101**	-6.779**

* Significant at 5% level

** Significant at 1% level

TABLE 12

RUN SET 4
COMPARISON OF PERCENT DIFFERENCES IN AVERAGE PERIOD COSTS
FOR THOSE SCHEME PAIRS WITH SIGNIFICANT t STATISTICS

CASE \ SCHEME PAIR	ML;ES	ML;EST	ML;SB	ML;AB	ES;EST	ES;SB	ES;AB	EST;SB	EST;AB	SB;AB
$f = .8;\ \lambda = 1+.05$	7.0	2.9	-0.9	-0.5	-3.9	-7.9	-7.5	-3.8	-3.4	0.3
$\lambda = 1+.1$	19.5	15.9	-1.4	-1.0	-3.0	-23.6	-20.6	-17.5	-17.0	0.4
$\lambda = 1+.5$	72.0	73.7	-2.3	-2.0	1.0	-76.0	-75.4	-77.8	-77.2	0.3
$f = .5:\ \lambda = 1+.05$	2.2	not sig.	not sig.	-0.4	-3.4	-0.2	-2.4	not sig.	not sig.	not sig.
$\lambda = 1+.1$	8.0	5.5	-0.5	-0.7	-2.2	-8.6	-8.8	-6.1	-6.2	not sig.
$\lambda = 1+.5$	22.4	22.2	-0.9	-0.9	not sig.	-23.4	-23.5	-23.2	-23.3	-0.0
$f = .3;\ \lambda = 1+.05$	7.3	not sig.	-0.4	-0.4	-4.5	-4.0	-4.0	not sig.	not sig.	not sig.
$\lambda = 1+.1$	13.2	9.6	-0.6	-0.6	-3.3	-13.9	-13.9	-10.3	-10.3	not sig.
$\lambda = 1+.5$	44.4	42.7	-1.2	-1.4	-1.2	-46.1	-46.4	-44.4	-44.6	-0.2

TABLE 13

RUN SET 5
SHOCKED DEMAND
f = .8, .5, .3; α = 1.; λ = 1↑5, 1↑10, 5↓1, 10↓1, 1↑5↓1
Average Cost Per Period

CASE \ SCHEME		ML		ES		EST		SB		AB	
		MEAN	S.D.	MEAN	S.D.	MEAN	S.D.	MEAN	S.D.	MEAN	S.D.
f = .8;	λ = 1↑5	8.282	0.234	6.032	0.134	6.101	0.154	8.316	0.236	8.294	0.236
	λ = 1↑10	17.914	0.256	10.150	0.131	10.172	0.163	17.995	0.265	17.981	0.265
	λ = 5↓1	7.115	0.144	5.876	0.146	6.074	0.147	7.114	0.143	7.107	0.143
	λ = 10↓1	12.674	0.132	10.031	0.175	10.291	0.158	12.675	0.132	12.696	0.132
	λ = 1↑5↓1	6.299	0.187	5.205	0.119	5.367	0.116	6.300	0.182	6.306	0.184
f = .5;	λ = 1↑5	4.992	0.121	4.484	0.100	4.567	0.105	4.996	0.122	4.996	0.122
	λ = 1↑10	9.394	0.119	7.650	0.102	7.731	0.095	9.402	0.120	9.405	0.120
	λ = 5↓1	5.444	0.114	4.547	0.103	4.616	0.99	5.445	0.113	5.446	0.114
	λ = 10↓1	10.334	0.097	8.117	0.123	8.097	0.124	10.336	0.096	10.332	0.097
	λ = 1↑5↓1	3.872	0.088	3.647	0.084	3.735	0.089	3.874	0.087	3.875	0.088
f = .3;	λ = 1↑5	10.546	0.238	8.932	0.219	9.352	0.216	10.556	0.241	10.539	0.236
	λ = 1↑10	19.159	0.258	14.001	0.209	14.372	0.274	19.204	0.256	19.196	0.243
	λ = 5↓1	14.234	0.371	9.719	0.264	9.726	0.382	14.251	0.372	14.239	0.372
	λ = 10↓1	30.715	0.366	17.701	0.479	17.436	0.504	30.724	0.364	30.659	0.377
	λ = 1↑5↓1	8.250	0.223	8.069	0.273	8.447	0.294	8.259	0.222	8.216	0.214

TABLE 14

RUN SET 5

T STATISTICS ($t_{.05}=2.093$, $t_{.01}=2.861$)

CASE \ SCHEME PAIR	ML;ES	ML;EST	ML;SB	ML;AB	ES;EST	ES;SB	ES;AB	EST;SB	EST;AB	SB;AB
f = .8; λ = 1↑5	13.363**	13.087**	-3.446**	-1.106	-1.105	-13.487**	-13.446**	-13.303**	-13.228**	2.308*
λ = 1↑10	38.146**	38,231**	-3.977**	-3.275**	-0.267	-37.561**	-37.651**	-37.140**	-37.718**	1.636
λ = 5↓1	13.811**	11.504**	1.0	1.064	-4.785**	-13.856**	-14.001**	-11.553**	-11.578**	0.865
λ = 10↓1	17.854**	14.488**	-1.0	-2.163*	-2.498*	-17.897**	-17.903**	-14.519**	-14.730**	-2.031
λ = 1↑5↓1	9.760**	7.700**	-0.102	-0.616	-2.883**	-9.755**	-9.726**	-7.710**	-7.701**	-1.561
f = .5; λ = 1↑5	9.454**	8.009**	-1.831	-1.831	-3.449**	-9.406**	-9.406**	-8.004**	-8.004**	-0.0
λ = 1↑10	34.310**	26.776**	-2.666*	-3.327**	-2.936**	-34.350**	-34.539**	-26.660**	-26.675**	-1.453
λ = 5↓1	14.634**	11.280**	-1.0	-0.809	-2.780*	-14.650**	-14.515**	-11.293**	-11.183**	-0.370
λ = 10↓1	23.578**	21.691**	-1.0	1.0	0.383	-23.646**	-23.590**	-21.875**	-21.730**	1.371
λ = 1↑5↓1	6.612**	3.323**	-0.370	-0.623	-3.621**	-6.711**	-6.864**	-3.353**	-3.468**	-0.370
f = .3; λ = 1↑5	8.452**	5.598**	-0.597	0.207	-3.197**	-8.355**	-8.596**	-5.539**	-5.448**	0.534
λ = 1↑10	25.476**	17.359**	-2.223*	-1.127	-3.683**	-25.296**	-26.238**	-17.309**	-17.399**	0.215
λ = 5↓1	12.986**	10.972**	-0.955	-0.140	-0.047	-13.249**	-12.985**	-11.270**	-10.988**	0.331
λ = 10↓1	24.286**	23.952**	-1.0	2.287*	1.129	-24.346**	-23.955**	-24.036**	-23.569**	2.188
λ = 1↑5↓1	0.837	-0.849	-0.960	.097	-2.801*	-0.880	-0.726	0.804	1.000	1.263

* Significant at 5% level

** Significant at 1% level

TABLE 15

RUN SET 5
COMPARISON OF PERCENT DIFFERENCES IN AVERAGE PERIOD COSTS
FOR THOSE SCHEME PAIRS WITH SIGNIFICANT t STATISTICS

CASE \ SCHEME PAIR	ML;ES	ML;EST	ES;EST	ES;SB	ES;AB	EST;SB	EST;AB
f = .8; λ = 1↑5	37.3	35.7	not sig.	-37.9	-37.5	-36.3	-36.0
λ = 1↑10	76.5	76.1	not sig.	-77.3	-77.2	-76.9	-76.8
λ = 5↓1	21.1	17.1	-3.4	-21.1	-20.9	-17.1	-17.0
λ = 10↓1	26.3	23.2	-0.3	-26.3	-26.6	-16.4	-23.4
λ = 1↑5↓1	21.0	17.1	-3.1	-21.0	-21.2	-17.4	-17.5
f = .5; λ = 1↑5	11.3	9.3	-1.9	-11.4	-11.4	-9.4	-9.4
λ = 1↑10	22.8	21.5	-1.1	-22.9	-22.9	-21.6	-21.7
λ = 5↓1	19.7	17.9	-1.5	-19.7	-19.8	-17.9	-18.0
λ = 10↓1	27.3	27.6	not sig.	-27.3	-27.3	-27.6	-27.6
λ = 1↑5↓1	6.2	3.7	-2.4	-6.2	-6.2	-3.7	-3.7
f = .3; λ = 1↑5	18.1	12.8	-4.7	-18.2	-18.0	-12.9	-12.7
λ = 1↑10	36.8	31.5	-4.1	-37.2	-37.1	-31.8	-31.7
λ = 5↓1	46.5	46.3	not sig.	-46.6	-46.5	-46.5	-46.4
λ = 10↓1	73.5	76.2	not sig.	-73.6	-73.2	-76.8	-75.8
λ = 1↑5↓1	not sig.	not sig.	-4.7	not sig.	not sig.	not sig.	not sig.

TABLE 16

RUN SET 6

$f = 0.95$

CASE \ SCHEME	Average Cost per Period					Average Percent Difference*					
	ML	ES	EST	SB	AB	ML;ES/EST	ML;SB	ML;AB	ES/EST;SB	ES/EST;AB	SB;AB
$\lambda = 0.5$	2.339	2.589	---	2.295	2.299	-10.7	not sig	not sig	12.8	12.6	not sig
$\lambda = 1$	3.695	3.810	---	3.681	3.697	not sig	not sig	not sig	not sig	not sig	not sig
$\lambda = 10$	17.151	17.820	---	17.150	17.006	-3.9	not sig	not sig	3.9	4.8	not sig
$\lambda = 1 + .5$	98.790	---	29.624	103.422	101.819	233.5	-4.7	-3.1	-249.1	-243.1	1.6
$\lambda = 1 \uparrow 5$	17.890	8.465	---	17.984	17.854	113.4	not sig	not sig	-113.4	-110.9	not sig

*For pairs showing significant difference at the 1% level

TABLE 17

RUN SET 7

$\lambda = 0.5,\ f = 0.80$

CASE \ SCHEME	Average Cost per Period			Average Percent Difference		
	ML	SB	AB	ML;SB	ML;AB	SB;AB
$\theta = 0.05$	1.496	1.617	1.632	-8.0	-9.1	not sig
$\theta = 0.50$	1.496	1.496	1.496	not sig	not sig	not sig
$\theta = 5.0$	1.496	1.500	1.501	not sig	not sig	not sig

TABLE 18

RUN SET 8
No Initialization Periods

$f = 0.80$, $\lambda = 0.50$

CASE \ SCHEME	Average Cost Per Period			Average Percent Difference		
	ML	SB	AB	ML;SB	ML;AB	SB;AB
5 Periods: $\theta = 0.05$	2.200	2.040	2.060	7.8	6.8	-1.0
$\theta = 0.50$	1.855	1.650	1.650	12.4**	12.4**	0
$\theta = 5.0$	7.155	7.190	8.340	0.0	-16.6**	-16.0**
10 Periods: $\theta = 0.05$	1.948	2.177	2.187	-11.8	-12.3*	-0.5
$\theta = 0.50$	1.757	1.612	1.617	8.3**	8.7*	0.4
$\theta = 5.0$	5.400	5.437	6.437	0.1	-19.2**	-18.4**

* Significant at 5% level; $t_{.05} \doteq 2.021$

** Significant at 1% level; $t_{.01} \doteq 2.704$

TABLE 19

COMPARISON OF C.D.F. FOR SB,AB; n = 0

	SB,ML	AB
θ = 0.05		
F(0)	0.952	0.952
F(1)	0.999	0.997
F(2)	1.000	0.999
S*(f=0.8)	0	0
θ = 0.50		
F(0)	0.607	0.667
F(1)	0.910	0.889
F(2)	0.986	0.968
F(3)	0.998	0.988
F(4)	1.000	0.996
F(5)	1.000	0.999
S*(f=0.8)	1	1
θ = 5		
F(0)	0.007	0.167
F(1)	0.040	0.306
F(2)	0.125	0.422
F(3)	0.268	0.518
F(4)	0.440	0.598
F(5)	0.616	0.665
F(6)	0.762	0.721
F(7)	0.867	0.768
F(8)	0.932	0.807
F(9)	0.968	0.839
S*(f=0.8)	7	8

rate) under each scheme.) As mentioned above, a comparison was made between the percent of difference between scheme pairs in Table 6. The figures in Table 6 were obtained for each scheme pair by subtracting the average period cost for the second scheme from the average period cost for the first scheme, and dividing the result by the smaller of the two average period costs. In comparing similar cases (e.g., $\lambda = 1$; $\alpha = 1$ and $\lambda = 1$; $\alpha = 3/4$) a larger percent difference generally results with $\alpha = 1$ than with $\alpha = 3/4$. This result does not occur in all comparisons between similar cases, but when it does not the difference is quite small. It is noted that when the case of trend is considered ($\lambda = 1+.5$), large percentage differences exist in scheme pairs AB;EST, ML;EST, and SB;EST. This is consistent with the results of run set 4.

Tables 7, 8 and 9 give the results of run set 3 for stationary demand, where $\alpha = 1$, and cost ratios of .8, .5 and .3 were used, when λ was varied from .5, 1, 5, 10 and 15. Table 7 shows mean average costs and standard deviations for 20 replications of 40 periods each. Table 8 displays the results of the paired t tests and Table 9 gives the percent difference for those cases in which significance was indicated by the paired t test.

No significant difference was indicated among the ML, SB, and AB schemes, schemes for which the theoretical basis is predicated on stationary. These schemes performed significantly better than ES and EST, although with the exception of the low mean demand ($\lambda = 0.5$ and $\lambda = 1$) cases, the average percent differences between ES and the "stationary" schemes were under 5%. It also appears that the critical cost ratio, f , has no particular effect on the percent differences. To check out a high value of f which in practice may be more realistic, the cases where $f = 0.95$, $\lambda = 0.5$, 1 and 10 respectively, were also investigated. The results are presented in Table 16 and are in line with the above statements, generally, with percent differences similar to the $f = 0.8$ cases. As expected, EST performed significantly poorer than ES on the stationary data; however, the percent differences are generally under 5% except when $\lambda = .5$. Even there, except for the very small f value, the percent differences were under 7%.

Thus, when considering long run average costs per period when some prior data exists, if the demand is stationary, one is better off using ML or the Bayes scheme especially for low demand, although using exponential smoothing for moderate or high demand is not too disadvantageous. (It should be noted that changing the smoothing constant(s) used in ES and EST might improve their performance. This effect was not studied here.)

The next situations investigated were those with a constant increasing long term trend. These results are presented in Tables 10, 11, and 12. Trends of 0.05, 0.1 and 0.5 per period were studied, starting with a base λ of 1. Although the t test indicated significance almost everywhere, when looking at the average percent differences, sizable values were found only between the "stationary" schemes (ML, SB, AB) and the exponential smoothing routines, with the exponential smoothing routines showing significantly better performance. (The maximum percent differences among ML, SB and AB were less than 3%.) For the trend of 0.5 per period, percent differences between ES and the ML, SB, AB schemes were in the 70's when $f = 0.8$. A run in which f was set at 0.95 shows percent differences in the 200's (see Table 16). It appears the smallest differences occur at $f = 0.5$, with differences becoming larger as f is increased or decreased, particularly for f increasing. One rather unexpected development did occur in this run set. In almost all cases EST did worse than ES which is certainly counter-intuitive. Only when $\lambda = 1 + 0.5$ for $f = 0.8$ was EST better. Although the t test showed significant differences, the average percent differences were small (always under 5%). It is believed the reason for this can be explained as follows. When using the Poisson distributions, unless the trend is quite sizable, the period to period variation is mistakenly picked up as trend in the EST scheme and thereby does more harm than good in adjusting the forecast. Figure 1 shows a plot of typically

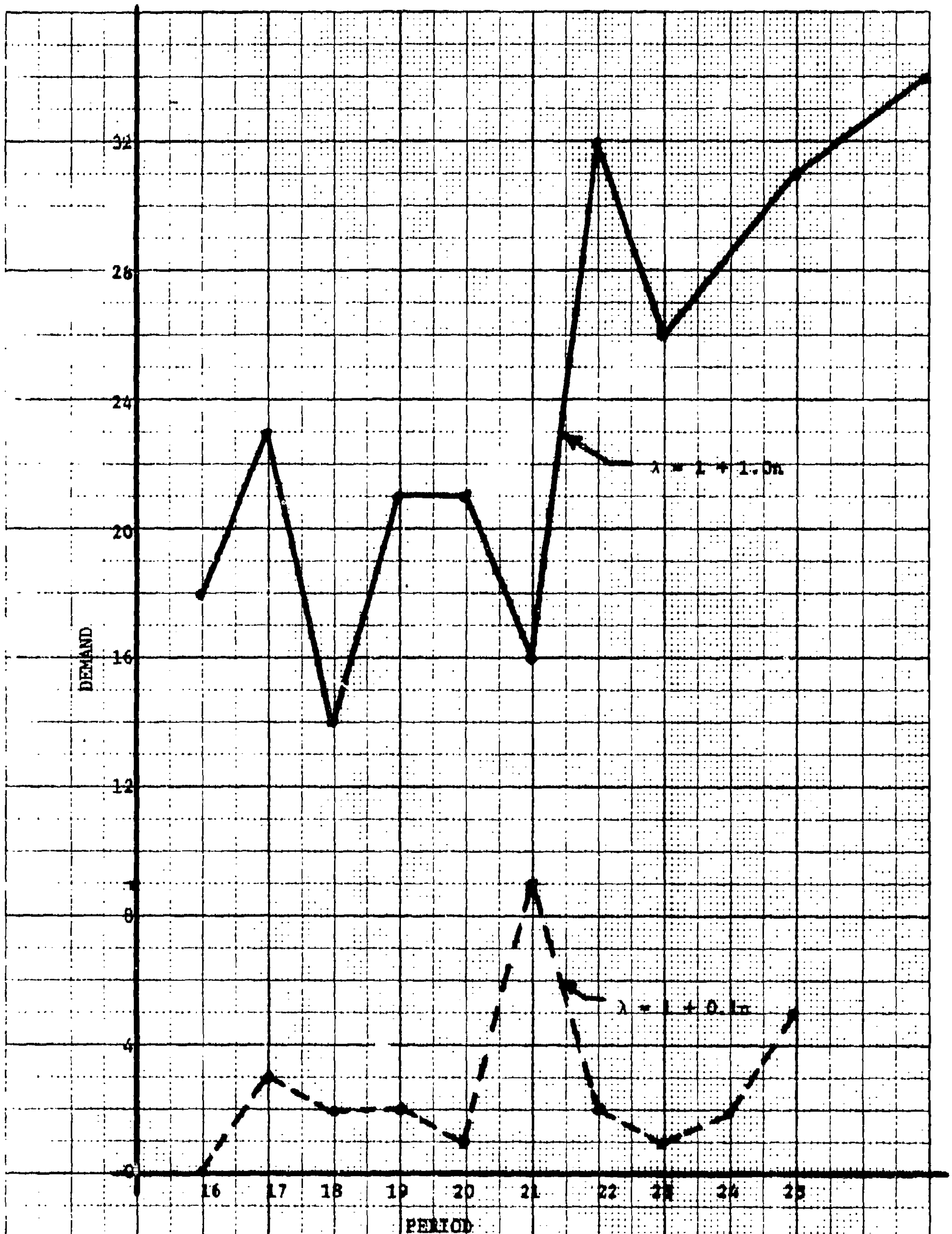

FIGURE 1 - SAMPLE DEMAND WITH TREND

generated data from a Poisson where the trend per period is 0.1 and 1.0 respectively, illustrating the period to period variation. For the 10 periods plotted, it is extremely difficult to spot trend when the trend is 0.1 per period.

In summing up for the trend cases, one is definitely better off using exponential smoothing rather than maximum likelihood or Bayes if a trend exists. Further, if the trend is sizable, EST is preferred; however, if the trend is small, EST performs slightly worse than ES.

The next set of runs investigated "shock" changes in demand. For $f = 0.8$, 0.5 and 0.3, λ was changed from 1 to 5, 1 to 10, 10 to 1, and 1 to 5 to 1 during the course of the runs (40 periods). Tables 13, 14 and 15 present the results. For each of the shock runs, the test periods are divided into two equal segments (except the last which is divided into 3) with the changes in λ taking place between segments.

Once again, although some t tests showed significance among ML, AB and SB, the greatest percent difference was found to be 0.45%; hence, it appears that there are no "practical" differences among these schemes. Thus, only percent differences between these schemes and the exponential smoothing schemes are given in Table 15. There appears to be a definite advantage in using ES or EST over AB, SB or ML when shock changes take place. Once again ES outperformed EST but all percent differences were under 5%. Results for $f = 0.95$ are shown in Table 16 and show that the differences are even greater for large f. Further, the $f = 0.95$ case shown in Table 16 along with those of Table 15 indicate that for large f values, the percent differences are more acute when the mean is shocked upward while for low f values the opposite is true. For the double shocked case, differences appear to be more pronounced for high f; however, had the double shock been 5 to 1 to 5, this situation may have been reversed.

The next series of runs were concerned with comparing ML, SB and AB for "small" mean demand under stationary conditions. Table 17 shows long range average costs for $\lambda = 0.5$ and $\theta = 0.05$, 0.5 and 5.0. Once again, as was shown in Table 2, the average cost rate is not very sensitive to θ, nor to which of the "stationary" schemes used (although if θ greatly underestimates λ, ML appears somewhat better). It does appear, however, it is preferable to overestimate the prior mean rather than underestimate it, when f is "high."

Up to now, all runs made had a pre-initialization period of 5 periods and an initialization period of 10 periods prior to considering costs. To investigate what happens in the initial periods without prior data, a set of runs was performed with no pre-initialization or initialization, run length of first 5 and then 10 periods, 40 replications and $\lambda = 0.5$ for $\theta = 0.05$, 0.5 and 5.0 respectively at $f = 0.8$. It is summarized in Table 18. We see that except where θ is much greater than λ ($\theta = 5$) there is essentially no difference between AB and SB. When $\theta = 5$, AB is the poorer. This is due to AB yielding a greater S^* in the first period ($S^*(AB) = 8$, $S^*(SB) = 7$; see Table 19). Since in most of the replications, the total demand over the entire 10 periods was less than 7, no stock outs ever occurred and AB always had a higher holding cost. Thus, it would appear it is not of any benefit to use AB over SB and might be a detriment.

Except for the $\theta = 5$ cases, however, the Bayes schemes seem to offer some advantage over ML when considering the first 5 periods. Significance was indicated only for the $\theta = 0.5$ case; however, for 10 periods, once again the Bayes schemes were significantly better for $\theta = 0.5$, but the percent differences were less than those for 5 periods. For $\theta = 0.05$, ML showed better, although results were not highly significant. Long run results (Run Set 3, Tables 7, 8 and 9 and Run Set 7, Table 17) showed no differences among ML, AB and SB for $\theta = 0.5$, but ML superior for $\theta = 0.05$. Thus, it appears that in the very short term, Bayes schemes may offer an advantage, especially if the prior mean is close to the true mean.

In order to better understand the similarities and differences among ML, SB and AB, formulas leading to the stockage levels, S* , for each scheme are presented below.

$$\text{ML:} \quad \sum_{x=0}^{S^*} (1/x!)\ e^{-\frac{t_n+\theta}{n+1}} \left(\frac{t_n + \theta}{n + 1}\right)^x = f$$

$$\text{SB:} \quad \sum_{x=0}^{S^*} (1/x!)\ e^{-\frac{t_n+1}{n+1/\theta}} \left(\frac{t_n + 1}{n + 1/\theta}\right)^x = f$$

$$\text{AB:} \quad \sum_{x=0}^{S^*} \binom{x + t_n}{x} \left(\frac{n + 1/\theta}{n + 1/\theta + 1}\right)^{t_n+1} \left(\frac{1}{n + 1/\theta + 1}\right)^x = f$$

Comparing ML and SB, we have already mentioned that as n gets large, then both converge to

$$\sum_{x=0}^{S^*} (1/x!)\ e^{-t_n/n}(t_n/n)^x = f\ .$$

Further, for $n = 0$, they both yield identical S* values. Also, if $\theta = 1$, they are identical. Their greatest difference would occur, then, for the early periods after the initial forecast and where θ is different from 1. Thus, in most cases, one would expect rather similar results from ML and SB, especially if results are considered over a long planning horizon.

In comparing SB and AB, it can be shown that for large n , these two schemes should produce similar results. Letting $m = n + 1/\theta + 1$ and $r = t_n + 1$, we can write

$$\text{AB:} \quad \sum_{x=0}^{S^*} \binom{x+r-1}{x} (1 - \tfrac{1}{m})^r (\tfrac{1}{m})^x = \sum_{x=0}^{S^*} g(x) = f\ .$$

The negative binomial $g(x)$ has a moment generating function (MGF) of

$$\text{MGF}(t) = \left(\frac{1 - \frac{1}{m}}{1 - \frac{1}{m}e^t}\right)^r .$$

Letting $\mu = r/m$, we have

$$MGF(t) = \frac{\left(1 - \frac{\mu}{r}\right)^r}{\left(1 - \frac{\mu e^t}{r}\right)^r} .$$

Letting $r \to \infty$ yields

$$MGF(t) = e^{-\mu}/e^{-\mu e^t} = e^{\mu(e^t - 1)}$$

which is the MGF for a Poisson, mean μ . When r is large, m must also be large which results only when n is large. Hence, for n large, the negative binomial goes to a Poisson with mean

$$\mu = r/m = \frac{t_n + 1}{n + 1/\theta + 1} .$$

Further, if n is large, then

$$\mu \doteq \frac{t_n + 1}{n + 1/\theta}$$

which is the mean for the Poisson when using scheme SB. Thus, when n is large, one would not expect very much difference among ML, SB and AB. For small n , AB will tend to produce higher S^* values, especially if θ is large. If θ is small, even for small n , AB and SB will tend to produce similar S^*'s . Table 19 compares the CDF's for $n = 0$, $\theta = 0.05$, 0.5 and 5. Only for $\theta = 5$, do different S^*'s result when $f = 0.8$.

VI. CONCLUSIONS

Keeping in mind the underlying assumptions and range of this study (namely that demand per period is Poisson with unknown mean λ and the cases studied generally cover λ's from 0.5 to 15), we summarize the conclusions as follows.

1. When considering long range average costs per period with some amount of prior data available, if the demand process is not stationary (either trend or shock change is present) exponential

smoothing shows superior performance over maximum likelihood or Bayes. If a sizable linear trend exists, EST shows up somewhat better than ES. Otherwise, ES is preferred.

2. If the demand process is stationary, the exponential smoothing schemes perform poorer than maximum likelihood and Bayes, especially for the lower values of mean demand ($\lambda = 0.5$ and 1). For higher λ's however, the differences were less than 5%.

3. If no knowledge is known about whether the demand process is likely to be stationary, exponential smoothing appears to be a good hedge against this type of uncertainty.

4. In no cases did there appear to be substantial differences among ML, SB and AB. Since ML is the simplest and most understood, it is recommended in the case of stationary demand with low mean.

5. When no prior data exists, and the initial period costs are considered for the case of stationary demand, SB appears to offer some advantage for the initial periods, the advantage being greater when θ is close to λ. Thus a possible strategy might be to use SB for the initial few periods then switch to ML.

Two further avenues of exploration would be of interest. First, a continuation of the initial period investigation for differences among ML, SB and AB for small mean stationary demands (even smaller than those studied here). It has been claimed that under these conditions, Bayes schemes should show superior performance. Although the results of Table 18 are to some degree inconclusive, they tend to indicate this might be so. Initial period studies of all schemes for nonstationary conditions would also be of interest.

Second, it would be enlightening to investigate the robustness of forecasting schemes if demand were not really Poisson. Schemes such as ES and ML do not have to be tied to any assumption concerning distribution of demand. Often in practice, what is done is to merely assume that forecast errors are normally distributed (this may or may not be true), keep a running calculation of the sample standard deviation of the forecast errors and use, then, in Equation (1) a normal CDF with mean equal to the forecast value and standard deviation equal to the sample standard deviation of forecast errors. It would be interesting to investigate this procedure under varying conditions.

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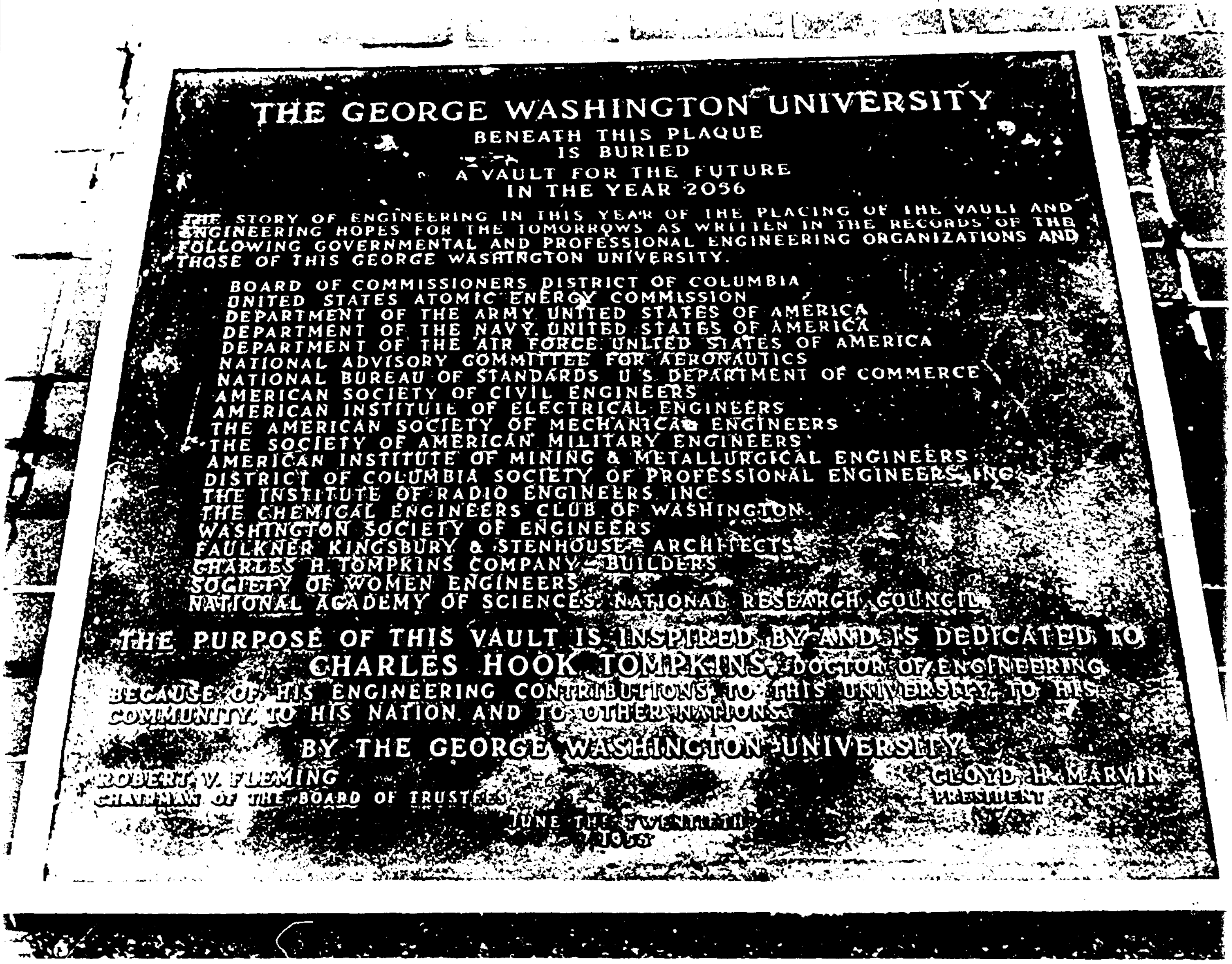

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